Existence of a multiplicative basis for a finitely spaced module over an aggregate

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It was proved in [1] that a finite-dimensional algebra, having finitely many isoclasses of indecomposable representations, admits a multiplicative basis. In [2] (Sections 4.10-4.12) an analogous hypothesis was formulated for finitely spaced modules over an aggregate and an approach to its proof was proposed. Our objective is to prove this hypothesis. Throughout this paper, k denotes an algebraically closed field.

Let us recall some definitions from [2] (see also [3]).

By definition, an *aggregate* $\mathcal{A}$ over k is a category that satisfies the following conditions:

a. For each $X, Y \in \mathcal{A}$, the set $\mathcal{A}(X, Y)$ is a finite-dimensional vector space over k;

b. The composition maps are bilinear;

c. $\mathcal{A}$ has finite direct sums;

d. Each idempotent $e \in \mathcal{A}(X, X)$ has the kernel.

As a consequence, each $X \in \mathcal{A}$ is a finite sum of indecomposables and the algebra of endomorphisms of each indecomposable is local.

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We denote by $\mathcal{JA}$ a spectroid of $\mathcal{A}$, i.e. a full subcategory formed by chosen representatives of the isoclasses of indecomposables, and let $\mathcal{R}_\mathcal{A}$ be the radical of $\mathcal{A}$. We suppose that $\mathcal{JA}$ has finitely many objects. For each $a, b \in \mathcal{JA}$, the space $\mathcal{R}_\mathcal{A}(a,b)$ consists of all irreversible morphisms of $\mathcal{A}(a,b)$, therefore, $\mathcal{A}(a,b) = \mathcal{R}_\mathcal{A}(a,b)$ for $a \neq b$. $\mathcal{A}(a,a) = k\ 1_a \oplus_k \mathcal{R}_\mathcal{A}(a,a)$.

A *module* M over an aggregate $\mathcal{A}$ consists of finite-dimensional vector spaces $M(X)$, one for each object $X \in \mathcal{A}$, and of linear maps $M(f) : M(X) \to M(Y)$, $m \mapsto fm$, $\quad f \in \mathcal{A}(X,Y)$, which satisfy the standard axioms: $1_X m = m, (f+g)m = fm + gm, (gf)m = g(fm), f(\alpha m) = \alpha(fm) = (\alpha f)m, \alpha \in k$. It gives a k-linear functor from $\mathcal{A}$ into the category mod k of finite-dimensional vector spaces over k. A module M over $\mathcal{A}$ is $faithful$ if $M(f) \neq 0$ for each nonzero $f \in \mathcal{A}(X,Y)$.

Define the *basis* of $(M, \mathcal{A})$ as a set $\{m_i^a, f_l^{ba}\}$ consisting of bases $m_1^a, m_2^a, \ldots$ of the spaces $M(a), a \in \mathcal{JA}$, and bases $f_1^{ba}, f_2^{ba}, \ldots$ of the spaces $\mathcal{R}_\mathcal{A}(a,b), a, b \in \mathcal{JA}$. The maximal rank of $M(f_l^{ba})$ is called the *rank* of a basis. A basis is called a *scalarly multiplicative basis* if it satisfies the following conditions:

a) Each morphism f_l^{ba} is *thin*, i.e. $f_l^{ba} = g + h$ implies rank $M(f_l^{ba}) \leq$ rank $M(g)$ and rank $M(f_l^{ba}) \leq$rank $M(h)$ for all $g, h \in \mathcal{A}(a,b)$;

b) Each product $f_l^{ba} m_i^a$ has the form $\lambda m_p^b, \lambda \in k$;

c) $f_l^{ba} m_i^a = \lambda m_p^b$, $f_l^{ba} m_j^a = \mu m_p^b$, and $\lambda, \mu \in k \setminus \{0\}$ imply $i = j$.

We say that the basis is *multiplicative* if each nonzero product $f_l^{ba} m_i^a$ is a basis vector m_p^b.

We denote by M^k the aggregate formed by all triples (V, h, X), where $V \in$ mod k, $X \in \mathcal{A}$, and $h \in \mathrm{Hom}_k(V, M(X))$. A morphism from (V, h, X) to (V', h', X') is defined by the pair of morphisms $\varphi \in \mathrm{Hom}_k(V, V')$ and $\xi \in \mathcal{A}(X, X')$ such that $h'\varphi = M(\xi)h$. We call these triples *spaces* on M. We say that M is *finitely spaced* if M^k has a finite spectroid.

The objective of the paper is to prove the following theorem:

Theorem. *If M is a faithful finitely spaced module over an aggregate $\mathcal{A}$, then $(M, \mathcal{A})$ admits a multiplicative basis of* rank ≤ 2.

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1. Construction of a scalarly multiplicative basis.

In Sections 1-3, M always denotes a finitely spaced module over an aggregate $\mathcal{A}$.

As shown in [2] (sections 4.7, 4.8), for each $a \in \mathcal{JA}$, the space $M(a)$ has a dimension $d(a) \le 3$ and a sequence $m_1, m_2, ..., m_{d(a)}$, where

$$m_i \in (\mathcal{R}_\mathcal{A}(a,a))^{i-1} M(a) \setminus (\mathcal{R}_\mathcal{A}(a,a))^i M(a),$$

is a basis of $M(a)$. It will be called a *triangular basis* because the matrix of each map $M(f), f \in \mathcal{A}(a,a)$, has a lower triangular form. We assume that each basis $m_1^a, ..., m_{d(a)}^a$ in a scalarly multiplicative basis is triangular (it is always triangular up to permutations of vectors).

A scalarly multiplicative basis is called *normed* if it satisfies the following condition:

d) $f_l^{ba} m_i^a = \lambda m_p^b$ and $\lambda \notin \{0,1\}$ imply that $f_l^{ba} m_{i'}^a = \lambda m_{p'}^b$ for some $i' < i$.

A scalarly multiplicative basis can be reduced to a normed basis by means of multiplication of f_l^{ba} by scalars.

A scalarly multiplicative basis is called *reduced* if it satisfies condition d) and the following condition:

e) if a morphism $\varphi = \sum_l \lambda_l f_l^{ba}$ is a product of basis morphisms, then

$$\text{rank } M(\varphi) = \sum_{\lambda_l \neq 0} \text{rank } M(f_l^{ba}).$$

At the end of this section, we shall prove that every multiplicative basis of $(M, \mathcal{A})$ is reduced if char $k \neq 2$.

Let $m_1^a, ..., m_{d(a)}^a$ be a fixed triangular basis of $M(a)$ for each $a \in \mathcal{JA}$. For m_j^a and m_i^b, we define a linear map $e_{ij}^{ba} : M(a) \to M(b)$ such that $e_{ij}^{ba} m_j^a = m_i^b$ and $e_{ij}^{ba} m_{j'}^a = 0$ for all $j' \neq j$.

Let $f \in \mathcal{R}_\mathcal{A}(a,b), a, b \in \mathcal{JA}$. We say that f is a *short morphism* if $f \notin \mathcal{R}_\mathcal{A}(c,b)\mathcal{R}_\mathcal{A}(a,c)$ for all $c \in \mathcal{JA}$, f is a *prime morphism* if $M(f) = e_{ij}^{ba}$, and f is a *double morphism* if

$$M(f) = e_{ij}^{ba} + \lambda e_{i'j'}^{ba}, e_{ij}^{ba} \notin M(a,b), i < i', j < j', 0 \neq \lambda \in k.$$

The coefficient λ is called the *parameter* of a double morphism.

Proposition 1. *A set $\{m_i^a, f_l^{ba}\}$ is a normed (reduced, respectively) scalarly multiplicative basis if and only if the following conditions are satisfied:*

1) *$m_1^a, m_2^a, ...$ is a triangular basis of $M(a), a \in \mathcal{JA}$,*

2) *$f_1^{ba}, f_2^{ba}, ...$ is the set of all prime and double morphisms of $\mathcal{A}(a,b)$, $a, b \in \mathcal{JA}$, except a single double morphism (a single short double morphism,*

respectively) if the number of double morphisms is equal to 3. Moreover, the number of double morphisms of $\mathcal{A}(a,b)$ is equal 0, 1 or 3, and, in the last case, there exists a short double morphism.

The statement of Proposition 1 about a normed scalarly multiplicative basis follows from Lemmas 1 and 5. The complete proof of Proposition 1 will be given in Section 3.

Lemma 1. *If $d(a)=2$, then $M(a,a)=k1_{M(a)}+ke^{aa}_{21}$. If $d(a)=3$, then $M(a,a)=k1_{M(a)}+ke^{aa}_{21}+ke^{aa}_{32}$ or*

$$M(a,a)=k1_{M(a)}+k(e^{aa}_{21}+\lambda_{aa}e^{aa}_{32})+ke^{aa}_{31} \tag{1}$$

and $0\neq\lambda_{aa}\in k$.

The proof of Lemma 1 is obvious.

For every linear map $\varphi: M(a)\to M(b)$, we denote by $\varphi_{ij}\in ke^{ba}_{ij}$ linear maps such that $\varphi=\sum\varphi_{ij}$. We introduce an order relation on $\{1,2,...,d(b)\}\times\{1,2,...,d(a)\}$ by $(i,j)\geq(l,r)$ if $i\leq l$ and $j\geq r$. A pair (l,r) is called a *step* of $\varphi\in M(a,b)$ if $\varphi_{lr}\neq 0$ and $\varphi_{ij}=0$ for all $(i,j)>(l,r)$. A pair (l,r) is called a *step* of $M(a,b)$ if $\psi_{lr}\neq 0$ for some $\psi\in M(a,b)$ and $\varphi_{ij}=0$ for all $\varphi\in M(a,b)$ and all $(i,j)>(l,r)$ ($l\geq r$ because each basis $m^a_1,m^a_2,...$ is a triangular).

Lemma 2. *If $a,b\in\mathcal{JA}$,$a\neq b,d(a)=d(b)=3$, and $M(a,b)$ has two steps (1,2) and (2,3), then $M(b,a)=ke^{ab}_{31}$.*

Proof. Let $\psi\in M(b,a)$. There is $\varphi\in M(a,b)$ having the steps (1,2) and (2,3). By Lemma 1, there exist $\epsilon\in M(a,a)$ and $\delta\in M(b,b)$ such that $\varphi'=\varphi\epsilon+\delta\varphi$ has the steps (1,1), (2,2) and (3,3). The inclusion $\mathcal{A}(b,a)\mathcal{A}(a,b)\subset\mathcal{R}_{\mathcal{A}}(a,a)$ implies $M(b,a)M(a,b)\subset M(\mathcal{R}_{\mathcal{A}}(a,a))=ke^{aa}_{21}+ke^{aa}_{31}+ke^{aa}_{32}$. Since $\psi\varphi'\in M(\mathcal{R}_{\mathcal{A}}(a,a))$, all steps of ψ are not higher that (2,1) and (3,2). Since $\psi\varphi\in M(\mathcal{R}_{\mathcal{A}}(a,a))$, we have $\psi\in ke^{ab}_{31}$.

Therefore, $M(b,a)\subset ke^{ab}_{31}$. Assume that $M(b,a)=0$. Let us examine the space $\mathcal{H}_\lambda=(k^6,h_\lambda,a^2\oplus b^2)\in M^k$, where $k^6=k\oplus k\oplus k\oplus k\oplus k\oplus k$, $a^2=a\oplus a$, $b^2=b\oplus b$, $\lambda\in k$, and h_λ is the linear mapping of k^6 into $M(a^2\oplus b^2)=(km^a_1)^2\oplus(km^a_2)^2\oplus(km^a_3)^2\oplus(km^b_1)^2\oplus(km^b_2)^2\oplus(km^b_3)^2$ with the matrix

$$\begin{pmatrix}1\\0\end{pmatrix}\oplus\begin{pmatrix}0\\1\end{pmatrix}\oplus\left(\begin{array}{cc|cc}1&0&1&0\\0&1&0&1\end{array}\right)^T\oplus\begin{pmatrix}1\\1\end{pmatrix}\oplus\begin{pmatrix}1\\\lambda\end{pmatrix}.$$

We show that $\mathcal{H}_\lambda\not\simeq\mathcal{H}_\mu$ if $\lambda\neq\mu$. Let (φ,ξ) be an isomorphism $\mathcal{H}_\lambda\to\mathcal{H}_\mu$. The linear mapping $M(\xi)$ has the block matrix (K_{ij}), $i,j\leq 6$, where K_{ij} are

2×2-matrices. By $M(b,a) = 0$ and Lemma 1, we have $K_{ij} = 0$ if $i < j$. Evidently, $K_{11} = K_{22} = K_{33}$, $K_{44} = K_{55} = K_{66}$ and $K_{43} = 0$.

Since $h_\mu \varphi = M(\xi) h_\lambda$, the matrix of the nondegenerate mapping φ also has the block form $(\Phi_{ij}), i, j \le 5$, where the blocks Φ_{11}, Φ_{22}, Φ_{44} and Φ_{55} are $1{\times}1$-matrices, the block Φ_{33} is a $2{\times}2$-matrix, and $\Phi_{ij} = 0$ if $i < j$. Moreover,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Phi_{11} = K_{11} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Phi_{22} = K_{22} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right)^T \Phi_{33} = (K_{33} \oplus K_{44}) \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right)^T,$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Phi_{44} = K_{55} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ \mu \end{pmatrix} \Phi_{55} = K_{66} \begin{pmatrix} 1 \\ \lambda \end{pmatrix}.$$

By the third equality, we obtain $K_{33} = K_{44}$, by the first and second equalities, we get

$$K_{11} = K_{22} = \dots = K_{66} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},$$

and, by the forth and fifth equalities, $\alpha = \beta$ and $\lambda = \mu$. We have infinitely many nonisomorphic indecomposable spaces $\mathcal{H}_\lambda, \lambda \in k$, on M. This proves Lemma 2.

Let $(l_1, r_1), \dots, (l_t, r_t)$ be all steps of $M(a,b)$. Set $S(a,b) = \sum_{(i,j)} k e_{ij}^{ba}$ (resp. $\overline{S}(a,b) = \sum_{(i,j)} k e_{ij}^{ba}$), where the sum is taken over all (i,j) such that there exists a step $(l_p, r_p) > (i,j)$ ($(l_p, r_p) \ge (i,j)$, respectively).

Lemma 3. *Let $a \ne b$ and $M(a,b)$ have the steps $(1,1)$, $(2,2)$ and $(3,3)$. Then there is no $\psi \in M(a,b)$ such that $M(a,b) = k\psi + S(a,b)$.*

Proof. Assume that there exists $\psi \in M(a,b)$ such that $M(a,b) = k\psi + S(a,b)$. By the form of $M(a,b)$ and $\mathcal{A}(b,a)\mathcal{A}(a,b) \subset \mathcal{R}_\mathcal{A}(a,a)$, we have $M(b,a) \subset k e_{21}^{ab} + k e_{31}^{ab} + k e_{32}^{ab}$.

Let us examine the space $\mathcal{H}_\lambda = (k^3, h_\lambda, a \oplus b)$, where $\lambda \in k$ and h_λ is the linear map from k^3 into

$$M(a \oplus b) = k m_1^a \oplus k m_2^a \oplus k m_3^a \oplus k m_1^b \oplus k m_2^b \oplus k m_3^b$$

with the matrix

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \lambda \end{array}\right)^T.$$

Let (φ, ξ) be an isomorphism $\mathcal{H}_\lambda \to \mathcal{H}_\mu$. It follows from the conditions imposed on $M(a,a), M(a,b), M(b,a)$ and $M(b,b)$ that the matrix of $M(\xi)$ has form

$$\left(\begin{array}{ccc|ccc} \alpha_1 & 0 & 0 & 0 & 0 & 0 \\ \alpha_2 & \alpha_1 & 0 & \gamma_1 & 0 & 0 \\ \alpha_4 & \alpha_3 & \alpha_1 & \gamma_3 & \gamma_2 & 0 \\ \hline \delta_1 & 0 & 0 & \beta_1 & 0 & 0 \\ \delta_4 & \delta_2 & 0 & \beta_2 & \beta_1 & 0 \\ \delta_6 & \delta_5 & \delta_3 & \beta_4 & \beta_3 & \beta_1 \end{array}\right).$$

Moreover, $\delta_1 = \delta\epsilon_1$, $\delta_2 = \delta\epsilon_2$, and $\delta_3 = \delta\epsilon_3$, where $\delta \in k$ and ϵ_1,ϵ_2, and ϵ_3 are the diagonal elements of the lower triangular matrix of ψ. By $h_\mu\varphi = M(\xi)h_\lambda$, we find successively that $\delta = 0$, the mapping φ has the lower triangular matrix with the diagonal $(\alpha_1, \alpha_1, \alpha_1)$, $\alpha_1 = \beta_1$, and $\lambda = \mu$.

Hence $\mathcal{H}_\lambda \not\simeq \mathcal{H}_\mu$ for $\lambda \neq \mu$ and M is infinitely spaced. We arrive at a contradiction that proves Lemma 3.

Lemma 4. $S(a,b) \subset M(a,b)$.

Proof. We must show that if (l,r) is a step of $M(a,b)$, then

$$S_{lr}(a,b) = \sum_{(i,j)<(l,r)} ke^{ba}_{ij} \subset M(a,b).$$

By Lemma 3, there exists a $\psi \in M(a,b)$ having the step (l,r) but not more then two steps. If ψ and $M(a,b)$ have the steps $(1,2)$ and $(2,3)$, then, by Lemma 2, $e^{ab}_{31}\psi \in M(a,a)$ has the unique step $(3,2)$. Hence,

$$M(a,a) = k1_{M(a)} \oplus ke^{aa}_{21} \oplus ke^{aa}_{31} \oplus ke^{aa}_{32}.$$

In all other cases, by Lemma 1, $S_{lr}(a,b)$ is contained in the space generated by all $\delta\psi\epsilon$, where $\epsilon \in M(a,a)$ and $\delta \in M(b,b)$. This proves Lemma 4.

By Lemma 4, we have the following lemma.

Lemma 5. *Let $a, b \in \mathcal{JA}$,$a \neq b$, and $M(a,b) \neq \overline{S}(a,b)$. Then only three cases can occur* $(\lambda_{ab} \neq 0 \neq \mu_{ab})$:

a) $M(a,b)$ has two steps (l_1, r_1) and (l_2, r_2), $l_1 < l_2$, and is equal to

$$k(e^{ba}_{l_1 r_1} + \lambda_{ab} e^{ba}_{l_2 r_2}) \oplus S(a,b);$$

b) $M(a,b)$ has the steps $(1,1), (2,2)$ *and* $(3,3)$ *and is equal to*

$$k(e^{ba}_{11} + \lambda_{ab} e^{ba}_{22}) \oplus ke^{ba}_{33} \oplus S(a,b),$$

or

$$k(e_{11}^{ba} + \lambda_{ab} e_{33}^{ba}) \oplus k e_{22}^{ba} \oplus S(a,b),$$

or

$$k(e_{22}^{ba} + \lambda_{ab} e_{33}^{ba}) \oplus k e_{11}^{ba} \oplus S(a,b);$$

c) $M(a,b)$ has the steps $(1,1), (2,2)$ and $(3,3)$ and is equal to

$$k(e_{11}^{ba} + \lambda_{ab} e_{22}^{ba}) \oplus k(e_{11}^{ba} + \mu_{ab} e_{33}^{ba}) \oplus S(a,b).$$

Remarks. 1) In a normed scalarly multiplicative basis, each long double morphism $\varphi \in \mathcal{A}(a,b)$ is the product of double basis morphisms. Indeed, let $\varphi = \tau\psi$, where $\psi \in \mathcal{R}_{\mathcal{A}}(a,c)$ and $\tau \in \mathcal{R}_{\mathcal{A}}(c,b)$. Then ψ is the unique double morphism of $\mathcal{A}(a,c)$ (otherwise, φ is the sum of prime morphisms). Therefore, ψ is a basis morphism. Similarly, τ is also a basis morphism.

2) A normed scalarly multiplicative basis is reduced if and only if all long double morphisms are basis morphisms. Indeed, let a long double morphism $\varphi \in \mathcal{A}(a,b)$ be not a basis morphism. Then $\mathcal{A}(a,b)$ has two double morphisms and φ is their linear combination. But this contradicts the definition of a reduced basis.

3) Lemma 1 and Lemma 5 imply the statement of Proposition 1 about a normed scalarly multiplicative basis. By Remark 2, to complete the proof of Proposition 1 we must prove that each $\mathcal{A}(a,b)$ $(a, b \in \mathcal{J}\mathcal{A})$ does not contain three long double morphisms.

4) If char $k \neq 2$, then every multiplicative basis is reduced. Indeed, otherwise, there is, by Remark 2, a long double morphism $\varphi \in \mathcal{A}(a,b)$, which is not a basis morphism. By Lemma 5, $\varphi = \psi - \tau$, where ψ and τ are basis long double morphisms of $\mathcal{A}(a,b)\varphi$ is a product of basis morphisms; hence $M(\varphi) = e_{ii}^{ba} + e_{jj}^{ba}$ and char $k = 2$.

2. The graph of a scalarly multiplicative basis.

In this section, we study some properties of a scalarly multiplicative basis and give the proof of Proposition 1.

Following [2] (Section 4.9), we define a poset $\mathcal{P}$, whose elements are the spaces $a_i = (\mathcal{R}_{\mathcal{A}}(a,a))^{i-1}M(a)$ $(a \in \mathcal{J}\mathcal{A},\ 1 \leq i \leq d(a))$ and where $a_i \leq b_j$ if and only if $\mathcal{A}(b,b) f a_i = b_j$ for some $f \in \mathcal{A}(a,b)$. The elements $a_i \in \mathcal{P}$ are in a one-to-one correspondence with the basis vectors m_i^a of every scalarly multiplicative basis $\{m_i^a, f_l^{ba}\}$, moreover, $a_i < b_j$ if and only if $f_l^{ba} m_i^a = \lambda m_j^a$ for some f_l^{ba} and $0 \neq \lambda \in k$. We decompose the poset $\mathcal{P}$ into disjoint totally

ordered subsets $\{a_1, ..., a_{d(a)}\}$, $(a_1 < a_2 < ... < a_{d(a)}, d(a) \leq 3)$; each of them is called a *double* if $d(a) = 2$ and a *triple* if $d(a) = 3$.

The following three lemmas were given in [2] without proofs.

Lemma 6 (see [2] (Lemma 4.12.1)). *The union* $\cup\{a_1, a_2, a_3\}$ *of all triples is totally ordered.*

Proof. The elements of a triple are totally ordered.

Let $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ be triples and let some a_i be not comparable with some b_j. We shall construct indecomposable spaces $\mathcal{H}_\lambda = (k^6, h_\lambda, a^2 \oplus b^2)$ on M, $\lambda \in k$, such that $\mathcal{H}_\lambda \not\simeq \mathcal{H}_\mu$ for $\lambda \neq \mu$.

For $i = 3$ and $j = 1$, the spaces $\mathcal{H}_\lambda$ were constructed in the proof of Lemma 2. For arbitrary i and j, $\mathcal{H}_\lambda$ is constructed analogously with the block

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right)^T$$

of $h_\lambda : k^6 \to M(a^2 \oplus b^2)$ located in the rows of

$$km_i^a \oplus km_i^a \oplus km_j^b \oplus km_j^b \subset M(a^2 \oplus b^2).$$

Let $(\varphi, \xi) : \mathcal{H}_\lambda \tilde{\to} \mathcal{H}_\mu$ and let (M_{ij}) be the block matrix of $M(\xi)$. Then (M_{ij}) is not upper block-triangular, but we can reduce (M_{ij}) to the upper block-triangular form by means of simultaneous transpositions of vertical and horizontal stripes, since the set $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ is partially ordered. Hence, M is infinitely spaced. We arrive at a contradiction that proves Lemma 6.

Lemma 7 (see [2] (Lemma 4.9)). *There are no elements* a_i, $a_{i'}$, b_j, *and* $b_{j'}$ *such that* $a_i \neq a_{i'}$, $b_j \neq b_{j'}$, a_i *is not comparable to* $b_{j'}$, *and* b_j *is not comparable to* $a_{i'}$. *There are no elements* a_i, $a_{i'}$, b_j, $b_{j'}$ c_l *and* $c_{l'}$ *such that* $a_i \neq a_{i'}$, $b_j \neq b_{j'}$, $c_l \neq c_{l'}$, a_i *is not comparable to* $b_{j'}$, b_j *is not comparable to* $c_{l'}$, *and* c_l *is not comparable to* $a_{i'}$.

Proof. In the first case, we set $\mathcal{H}_\lambda = (ke_1 \oplus ke_2, h_\lambda, a \oplus b) \in M^k$, where $h_\lambda e_1 = m_i^a + m_{j'}^b$ and $h_\lambda e_2 = m_j^b + \lambda m_{i'}^a$. In the second case, we set $\mathcal{H}_\lambda = (ke_1 \oplus ke_2 \oplus ke_3, h_\lambda, a \oplus b \oplus c)$, where $h_\lambda e_1 = m_i^a + m_{j'}^b$, $h_\lambda e_2 = m_j^b + m_{l'}^c$, and $h_\lambda e_3 = m_l^c + \lambda m_{i'}^a$. Obviously, $\mathcal{H}_\lambda \not\simeq \mathcal{H}_\mu$ for $\lambda \neq \mu$.

Lemma 8 (see [2] (Lemma 4.12.2)). *Each triple contains at least two elements comparable with all elements of all doubles.*

Proof. Assume that Lemma 8 is not true for a triple $\{a_1, a_2, a_3\}$ and doubles $\{b_1, b_2\}$ and $\{c_1, c_2\}$.

Case 1. Assume that $b \neq c$. For definiteness, we suppose that a_2 is not comparable to b_1 and a_3 is not comparable to c_1.

For each representation $\mathcal{H}$

$$\begin{array}{ccccccccc} & & & & k^{r_1} & & & & \\ & & & & A_1\downarrow & & & & \\ k^{r_4} \xrightarrow{B_2} & k^{t_2} & \xleftarrow{B_1} k^{r_2} \xrightarrow{A_2} & & k^{t_1} & \xleftarrow{A_3} k^{r_3} \xrightarrow{C_1} & k^{t_3} & \xleftarrow{C_2} k^{r_5} \end{array}$$

of the quiver $\tilde{E}_7$ (see [2], (Section 6.3)), we construct the space

$$\overline{\mathcal{H}} = (k^{r_1+\ldots+r_5}, h, a^{t_1} \oplus b^{t_2} \oplus c^{t_3}) \in M^k,$$

where

$$h = A_1 \oplus \begin{pmatrix} A_2 \\ B_1 \end{pmatrix} \oplus \begin{pmatrix} A_3 \\ C_1 \end{pmatrix} \oplus B_2 \oplus C_2$$

is a linear mapping of $k^{r_1+\ldots+r_5}$ into

$$M(a^{t_1} \oplus b^{t_2} \oplus c^{t_3}) = (km_1^a)^{t_1} \oplus \left[(km_2^a)^{t_1} \oplus (km_1^b)^{t_2}\right] \oplus \\ \left[(km_3^a)^{t_1} \oplus (km_1^c)^{t_3}\right] \oplus (km_2^b)^{t_2} \oplus (km_2^c)^{t_3}.$$

The functor $\mathcal{H} \mapsto \overline{\mathcal{H}}$ on the representations $\mathcal{H}$ with injective A_1, A_2, A_3, B_2, and C_2 preserves indecomposability and *heteromorphism* (i.e. $\mathcal{H} \simeq \mathcal{H}'$ if $\overline{\mathcal{H}} \simeq \overline{\mathcal{H}}'$). Indeed, let $(\varphi, \xi) : \overline{\mathcal{H}} \tilde{\to} \overline{\mathcal{H}}'$. The nondegenerate linear maps φ and $M(\xi)$ have the block forms (Φ_{ij}), $i, j \leq 5$, and (K_{ij}), $i, j \leq 7$. The equality $h'\varphi = M(\xi)h$ implies $A_1'\Phi_{11} = K_{11}A_1$,

$$\begin{pmatrix} A_2' \\ B_1' \end{pmatrix} \Phi_{22} = \begin{pmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{pmatrix} \begin{pmatrix} A_2 \\ B_1 \end{pmatrix}, \qquad B_2'\Phi_{44} = K_{66}B_2,$$

$$\begin{pmatrix} A_3' \\ C_1' \end{pmatrix} \Phi_{33} = \begin{pmatrix} K_{44} & K_{45} \\ K_{54} & K_{55} \end{pmatrix} \begin{pmatrix} A_3 \\ C_1 \end{pmatrix}, \qquad C_2'\Phi_{55} = K_{77}C_2.$$

Since $\{a_1, a_2, a_3\}$ is a triple and $\{b_1, b_2\}$ and $\{c_1, c_2\}$ are doubles, we have $K_{11} = K_{22} = K_{44}$, $K_{33} = K_{66}$, and $K_{55} = K_{77}$. Since a_2 is not comparable to b_1 and a_3 is not comparable to c_1, we have $K_{23} = 0$, $\mathrm{K}_{32} = 0$, $K_{45} = 0$, and $K_{54} = 0$. Hence, the diagonal blocks of (Φ_{ij}) and (K_{ij}) determine a morphism $\mathcal{H} \to \mathcal{H}'$.

We shall show that this morphism is an isomorphism, i.e. the diagonal blocks Φ_{ii} and K_{ii} are invertible. By strengthening the partial order relation in $\{a_1,\ a_2,\ a_3,\ b_1,\ b_2,\ c_1,\ c_2\}$, we obtain a total order relation $\ll$ such that $a_2 \ll b_1$ and $a_3 \ll c_1$ (these pairs are not comparable with respect to $<$).

We transpose the horizontal stripes of the matrices of h and h' according to the new order. Then we transpose the vertical stripes to get lower trapezoidal matrices. Correspondingly, we transpose the blocks of (Φ_{ij}) and (K_{ij}). Then the new matrix (K_{ij}) has a lower triangular form. The upper nonzero blocks of vertical stripes are the injective maps A_1, A_2, A_3, B_2, and C_2 (since $a_2 \ll b_1$ and $a_3 \ll c_1$). It follows from $h'\varphi = M(\xi)h$ that (Φ_{ij}) also has a lower triangular form. Hence, the diagonal blocks Φ_{ii} and K_{ii} are invertible and $\mathcal{H} \simeq \mathcal{H}'$.

But the quiver $\tilde{E}_7$ admits an infinite set of nonisomorphic indecomposable representations of the form $\mathcal{H}$ with injective A_1, A_2, A_3, B_2, and C_2 (and surjective B_1 and C_1, which will be used in the case 2). These representations are determined by the matrices

$$(A_1 \mid A_2 \mid A_3) = \left(\begin{array}{cc|ccc|ccc} 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right),$$

$$(B_1 \mid B_2) = (C_1 \mid C_2) = \left(\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right),$$

and they are nonisomorphic for different $\alpha \in k$. This contradicts the assumption that M is finitely spaced.

Case 2. Assume that $b = c$. By Lemma 7, if a_i is not comparable to b_1, and a_j is not comparable to b_2, then $i = j$. Let a_2 and a_3 be not comparable to b_1. Then $a_1 < b_1$ and $a_3 < b_2$.

As in the case 1, for each representation $\mathcal{H}$ of the quiver $\tilde{E}_7$ with injective A_1, A_2, A_3, B_2, and C_2 and surjective B_1 and C_1, we construct the space $\widehat{\mathcal{H}} = (k^{r_1+\dots+r_5}, h, a^{t_1} \oplus b^{t_2+t_3}) \in M^k$, where

$$h = A_1 \oplus \begin{pmatrix} A_2 \\ B_1 \end{pmatrix} \oplus \begin{pmatrix} A_3 \\ C_1 \end{pmatrix} \oplus B_2 \oplus C_2$$

is a linear mapping of $k^{r_1+\dots+r_5}$ into

$$M(a^{t_1} \oplus b^{t_2+t_3}) = (km_1^a)^{t_1} \oplus \left[(km_2^a)^{t_1} \oplus (km_1^b)^{t_2}\right] \oplus \\ \left[(km_3^a)^{t_1} \oplus (km_1^b)^{t_3}\right] \oplus (km_2^b)^{t_2} \oplus (km_2^b)^{t_3}.$$

Let $(\varphi, \xi) : \widehat{\mathcal{H}} \tilde{\to} \widehat{\mathcal{H}}'$. It follows from the order relation for $\{a_1, a_2, a_3, b_1, b_2\}$ that all blocks over the diagonal of the block matrix $K = (K_{ij})_{i,j=1,2,\dots,7}$ of

the mapping $M(\xi)$ are zero except the blocks $K_{35} = K_{67}$. Let us prove that they are zero, too.

Indeed, by comparing the blocks with index $(2,3)$ in the equality $h'\varphi = M(\xi)h$, we obtain $A_2'\Phi_{23} = 0$ and $\Phi_{23} = 0$ since A_2' is injective. By comparing the blocks with index $(3,3)$, we obtain $\mathrm{B}_1'\Phi_{23} = K_{35}C_1$ and $\mathrm{K}_{35} = 0$ since C_1 is surjective.

Hence K is the lower block-triangular matrix. Therefore Φ also is a lower block-triangular matrix, the diagonal blocks K_{ii} and Φ_{ii} of which are invertible, $\mathcal{H} \simeq \mathcal{H}'$. This proves our lemma.

Now fix a normed scalarly multiplicative basis $\{m_i^a, f_l^{ba}\}$ and define the oriented graph Γ, the set of vertices Γ_0 of which is the poset $\mathcal{P}$ and there is an arrow $a_p \to b_q$ $(a_p, b_q \in \Gamma_0)$ if and only if $M(f_l^{ba}) = \lambda e_{qp}^{ba} + \mu e_{q'p'}^{ba}$ for some short double morphism f_l^{ba} (then there is an arrow $a_{p'} \to b_{q'}$ and we shall say that the arrows $a_p \to b_q$ and $a_{p'} \to b_{q'}$ are *connected*). An arrow $a_p \to b_q$ will be called a *weak arrow* if $\mathcal{A}(a,b)$ contains three double morphisms. Each weak arrow is connected with two arrows. The others will be called *strong arrows*, each of them is connected exactly with one arrow.

Lemma 9. *Let $a_i < b_j < c_r$ and $a_i \to c_r$ be an arrow. Then $a \neq b \neq c \neq a$, $i = r$, the spaces $\mathcal{A}(a,b)$, $\mathcal{A}(b,c)$ and $\mathcal{A}(a,c)$ contain exactly* 1, 1 *and* 3 *double morphisms respectively, and there exists a pair of oriented paths $(a_i \to \ldots \to b_j \to \ldots \to c_i,\ a_{i'} \to \ldots \to b_{j'} \to \ldots \to c_{i'})$ consisting of connected strong arrows, and a pair of connected weak arrows $(a_i \to c_i,\ a_{i''} \to c_{i''},\ i \neq i'')$. In the case of a reduced scalarly multiplicative basis, there is no other arrow from $\{a_l\}$ to $\{c_l\}$.*

Proof. Since $a_i < b_j < c_r$, there are morphisms $g \in \mathcal{A}(a,b)$ and $h \in \mathcal{A}(b,c)$ such that $M(g) = \alpha e_{ji}^{ba} + \beta e_{j'i'}^{ba}$ and $M(h) = \gamma e_{rj}^{cb} + \delta e_{r''j''}^{cb}$ $(\alpha, \beta, \gamma, \delta \in k$ and $\alpha \neq 0 \neq \gamma)$. If hg is a prime morphism, then $M(hg) = \alpha\gamma e_{ri}^{ca}$ contradicts the existence of the arrow $a_i \to c_r$. Hence hg is a double morphism, $\beta \neq 0 \neq \delta$, $j' = j''$ and g and h are the unique double morphisms of $\mathcal{A}(a,b)$ and $\mathcal{A}(b,c)$ respectively. The space $\mathcal{A}(a,c)$ contains the double morphism hg and the short double morphism corresponding to the arrow $a_i \to c_r$, hence $M(a,c)$ has the form from item c) of Lemma 5.

If the basis is reduced then by Remark 2 of Sect.1, the double morphism hg is a basis morphism and there is only one pair of connected arrows from $\{a_l\}$ to $\{c_l\}$. This proves our lemma.

Proof of Proposition 1. By Remark 3 of Sect.1, we must prove that each space $\mathcal{A}(a,c)$ $(a, c \in \mathcal{J}\mathcal{A})$ does not contain three long double morphisms.

By contradiction let $f_1, f_2, f_3 \in \mathcal{A}(a, c)$ be three long double morphisms and let $f_r = h_r g_r$, where g_r is a short double morphism and $r = 1, 2, 3$. The morphisms g_1, g_2 and g_3 correspond to the pairs of connected arrows ($a_1 \to x_i$, $a_2 \to x_{i'}$), ($a_1 \to y_j$, $a_3 \to y_{j'}$), and ($a_2 \to z_l$, $a_3 \to z_{l'}$).

Let $x_i < y_j$. By putting $(a_i, b_j, c_r) = (a_1, x_i, y_j)$ in Lemma 9, we obtain that $\mathcal{A}(a, y)$ contains three double morphisms. By putting $(a_i, b_j, c_r) = (a_1, y_j, c_1)$ in Lemma 9, we have that $\mathcal{A}(a, y)$ contains exactly one double morphism.

Hence x_i is not comparable to y_j. Similarly $x_{i'}$ is not comparable to z_l, and $\mathrm{y}_{j'}$ is not comparable to $\mathrm{z}_{l'}$. This contradicts Lemma 7 and proves Proposition 1.

We shall now assume that the graph Γ is obtained from a reduced scalarly multiplicative basis.

Lemma 10. *If two arrows start from (stop at) the same vertex, then the arrows connected with them start from (stop at) different vertices.*

Proof. By contradiction, let $b_j \leftarrow a_i \to c_r$ and $b_{j'} \leftarrow a_{i'} \to c_{r'}$ be connected arrows. If $b_j < c_r$, then $a_i < b_j < c_r$ and, by Lemma 9, the arrows connected with $a_i \to b_j$ and $a_i \to c_r$ must start from different vertices, but they start from $a_{i'}$. Analogously, $b_{j'}$ is not comparable to $c_{r'}$. This contradicts Lemma 7.

Lemma 11. *There are no two arrows starting from (stopping at) the same vertex of a double. There are no three arrows starting from (stopping at) the same vertex of a triple.*

The proof follows from Lemma 10.

Lemma 12. *There are at most two different pairs of connected arrows starting from (stopping at) the same triple.*

Proof. By contradiction, let there be three pairs of connected arrows from a triple $\{a_1, a_2, a_3\}$ to $\{b_i\}$, $\{c_i\}$, $\{d_i\}$. Since there exist at most two pairs of connected arrows from a triple to a triple, then there are no three coinciding objects among a, b, c, d. Hence there exist five possibilities up to a permutation of b, c, d: 1) $a = b \neq c \neq d$, $a \neq d$; 2) $a = b \neq c = d$; 3) $a \neq b = d \neq c$, $a \neq c$; 4) a, b, c, d are distinct and there are two arrows $a_i \to b_j$ and $a_i \to c_r$, $b_j < c_r$; 5) a, b, c, d are distinct and for each pair of arrows $a_i \to x$ and $a_i \to y$, the vertices x and y are incomparable.

By Lemmas 9-11, we have the following subgraphs of Γ in cases $1, 3$ and 4:

$$
1)\quad
\begin{tikzcd}[row sep=small, column sep=small]
 & d_{j'} \\
a_3 \arrow[ur] \arrow[dr] & \\
 & c_{i'} \\
 & d_j \\
a_2 \arrow[uuu] \arrow[ur] & \\
 & \\
a_1 \arrow[uu] \arrow[dr] & \\
 & c_i
\end{tikzcd}
\; ; \qquad
3)\quad
\begin{tikzcd}[row sep=small, column sep=small]
a_{i''} \arrow[rr] \arrow[dr] & & b_{i''} \\
 & c_{j'} & \\
a_{i'} \arrow[rr] \arrow[dr] & & b_{i'} \\
 & c_j & \\
a_i \arrow[rr, shift left] \arrow[rr, shift right] & & b_i
\end{tikzcd}
\; ; \qquad
4)\quad
\begin{tikzcd}[row sep=small, column sep=small]
a_{i''} \arrow[rrr] \arrow[dr] & & & c_{i''} \\
 & d_{r'} & & \\
 & b_{j'} \arrow[rr, dotted] & & c_{i'} \\
a_{i'} \arrow[ur] \arrow[dr] & & & \\
 & d_r & & \\
 & b_j \arrow[rr, dotted] & & c_i \\
a_i \arrow[ur] \arrow[urrr] & & &
\end{tikzcd}
\; .
$$

Consider these cases.

1) If $c_i < a_2$ or $d_j < a_3$, then by Lemma 9, $\mathcal{A}(a,a)$ contains three double morphisms, which is a contradiction. If $a_2 < c_i$ or $a_3 < d_j$, then by Lemma 9, there is an arrow $a_2 \to c_i$ or $a_3 \to d_j$, in contradiction with Lemma 11. Hence a_2 is incomparable with c_i and a_3 is incomparable with d_j, which is impossible by Lemma 8.

2) This case is similar to the previous one.

3) The inequality $b_{i'} < c_j$ is impossible, by Lemma 9, because $\mathcal{A}(a,b)$ contains three double morphisms. The inequality $b_{i'} > c_j$ is impossible, by Lemma 9, because there are four arrows from $\{a_l\}$ to $\{b_l\}$. Hence $b_{i'}$ is incomparable with c_j. Analogously $b_{i''}$ is not comparable with $c_{j'}$ in contradiction with Lemma 7.

4) The inequalities $c_{i'} < d_r$ and $c_{i''} < d_{r'}$ are impossible, by Lemma 9, because $\mathcal{A}(a,c)$ contains three double morphisms. If $d_r < c_{i'}$ or $d_{r'} < c_{i''}$, then the double morphism $\lambda e^{ca}_{i'i'} + \mu e^{ca}_{i''i''}$ $(\lambda \neq 0 \neq \mu)$ is a product of double morphisms in $\mathcal{A}(a,d)$ and $\mathcal{A}(d,c)$, hence $\mathcal{A}(a,c)$ contains two long double morphisms in contradiction with the arrows $a_i \to c_i$ and $a_{i''} \to c_{i''}$. Hence $c_{i'}$ is not comparable to d_r and $c_{i''}$ is not comparable to $d_{r'}$, in contradiction with Lemma 7.

5) This case is impossible by Lemma 7. The proof of Lemma 12 is thus complete.

3. A construction of a multiplicative basis.

In this section we shall prove the following proposition.

Proposition 2. *From every reduced scalarly multiplicative basis, we can obtain a reduced scalarly multiplicative basis by means of multiplications* of *the basis vectors by non-zero elements of k.*

Let Γ be the graph of a reduced scalarly multiplicative basis $\{m^a_i, f^{ba}_l\}$ and let Γ_1 be the set of its arrows. An integral function $z : \Gamma_1 \to \mathbb{Z}$ will be

called a *weight function* and its value at an arrow will be called the *weight of the arrow* if:

a) $z(\alpha_1) = -z(\alpha_2)$ for each pair of connected arrows α_1, α_2;

b) the sum of the weights of all arrows stopping at a vertex $v \in \Gamma_0$ is equal to the sum of the weights of all arrows starting from v (this sum will be called the *weight of* v and will be denoted by $z(v)$).

Lemma 13. *There exists no non-zero weight function.*

Proof. By contradiction let $z : \Gamma_1 \to \mathrm{Z}$ be a non-zero weight function. An arrow α will be called *nondegenerate* if $z(\alpha) \neq 0$.

Let $v_1 < ... < v_m$ be the set of all vertices of the triples of Γ. For each vertex v_i, we denote by $v_{i'}, v_{i''}$ the two vertices such that $\{v_i, v_{i'}, v_{i''}\}$ is a triple.

By an *elementary path of length s* we shall mean a sequence of arrows of the form

$$v_p \xrightarrow{\lambda_1} u_1 \xrightarrow{\lambda_2} u_2 \rightarrow \ldots \rightarrow u_{s-1} \xrightarrow{\lambda_s} v_q \,, \qquad (2)$$

where $u_1, ..., u_{s-1}$ are vertices of doubles (they may be absent, i.e. a path may consist of exactly one arrow) and $z(\lambda_1) \neq 0$. Then by Lemma 11 and item b) of the definition of a weight function, $z(\lambda_1) = z(\lambda_2) = ... = z(\lambda_s)$, this non-zero integer we shall call the *weight of path* (2). We shall say that the elementary path (2) *avoids* a vertex v_i if $p < i < q$. Now we establish some properties of elementary paths:

A. The intersection of two elementary paths does not contain any vertex of a double.

B. Each nondegenerate arrow is contained in an elementary path.

C. If a vertex v_i is avoided by an elementary path (2) having length at least 2, then the v_i is incomparable with some vertex u_l in this path. Otherwise, $v_p < u_1 < ... < u_{s-1} < v_q$ implies one of the following conditions: $v_p < v_i < u_1$ or $u_j < v_i < u_{j+1}$ for some j or $u_{s-1} < v_i < v_q$. This contradicts Lemma 9 because the vertices $u_1, ..., u_{s-1}$ are contained in doubles.

D. If a vertex of a triple is avoided by an elementary path of length at least 2, then all other vertices of this triple can not be avoided by any elementary path having length ≥ 2. This follows from property C and Lemma 8.

E. The sum of the weights of all elementary paths avoiding a vertex v_i is equal to $-z(v_i)$. Indeed, this is obvious for v_1 because, by property B, only arrows having weight 0 can stop at v_1. If property E is true for v_i, then the sum of the weights of all elementary paths avoiding v_i or starting from v_i is

equal to 0. But the set of these paths coincides with the set of all elementary paths avoiding v_{i+1} or stopping at v_{i+1}. Hence property E is true for v_{i+1}.

F. Let a triple $\{b_1, b_2, b_3\}$ satisfy the following conditions: 1) there is no nondegenerate arrow starting from $a < b_1$; 2) there is a pair of connected degenerate strong arrows starting from (b_1, b_2) or (b_1, b_3); 3) there is a pair of connected nondegenerate weak arrows starting from (b_2, b_3). Then there exists a triple $\{a_1, a_2, a_3\}$ satisfying the same conditions and $a_1 < b_1$. Indeed, let for definiteness the pair of connected degenerate strong arrows start from (b_1, b_2). From $z(b_1) = 0$, $z(b_2) = -z(b_3) \neq 0$ and properties D and E, it follows that b_2 and b_3 is avoided by a nondegenerate arrow. Let b_3 be avoided by a nondegenerate arrow $a_i \to c_j$. Then $a_i < b_3 < c_j$. By Lemma 9, there exists a path $a_i \to ... \to b_3 \to ... \to c_j$ consisting of strong arrows. But by Lemma 12, there is only a weak arrow starting from b_3. Hence b_2 is avoided by some nondegenerate arrow $a_i \to c_j$. By Lemma 9, it is a weak arrow, $i = j$ and there is a path $a_i \to ... \to b_2 \to ... \to c_i$ consisting of strong arrows. But there is only one strong arrow starting from b_2 and it is connected with an arrow starting from b_1. Hence the arrows connected with $a_i \to ... \to b_2 \to ... \to c_i$ compose the path $a_{i'} \to ... \to b_1 \to ... \to c_{i'}$. The triple $\{a_1, a_2, a_3\}$ satisfies our requirement.

Let c_l be the vertex such that there is a nondegenerate arrow starting from c_l and there is no nondegenerate arrow starting from $b < c_l$. Then there is no nondegenerate arrow stopping at c_l, hence $z(c_l) = 0$ and there are two arrows starting from c_l and having the weights n and $-n$, moreover, $l = 1$ and the arrows connected with them start from c_2 and c_3. Since $z(c_2) = -z(c_3) = \pm n \neq 0$, the vertices c_2 and c_3 are avoided by elementary paths, and one of them is a nondegenerate arrow. Let for definiteness c_2 be avoided by a nondegenerate arrow $b_i \to d_j$. By Lemma 9, $i = j$ and there is a path $\mathrm{b}_i \to ... \to c_2 \to ... \to d_i$. Since there exists exactly one arrow starting from c_2 and this arrow is connected with an arrow starting from c_1, we have that the arrows connected with $\mathrm{b}_i \to ... \to c_2 \to ... \to d_i$ compose the path $\mathrm{b}_{i'} \to ... \to c_1 \to ... \to d_{i'}$. Since $b_{i'} < c_1$, there is no nondegenerate arrow starting from $b_{i'}$. Hence the arrow $b_i \to d_i$ is connected with the arrow $b_{i''} \to d_{i''}$, where $i' \neq i''$ and $i' = 1$. By applying property F to the triple $\{b_1, b_2, b_3\}$, we obtain a triple $\{a_1, a_2, a_3\}$. By applying property F to the triple $\{a_1, a_2, a_3\}$, we obtain another triple and so on. This contradicts the finiteness of the graph Γ. This proves our Lemma.

Proof of Proposition 2. We number all vertices and all arrows of the

graph Γ :

$$\Gamma_0 = \{a_1, a_2, ..., a_r\}, \qquad \Gamma_1 = \{f_{11}, f_{12}, ..., f_{s1}, f_{s2}\}.$$

where $f_{j1} : a_{p(j1)} \to a_{q(j1)}$ and $f_{j2} : a_{p(j2)} \to a_{q(j2)}$ are two connected arrows and $a_{p(j1)} < a_{p(j2)}$. Let the basis vector m_i correspond to the vertex a_i and let the double morphism f_i correspond to the pair (f_{j1}, f_{j2}). Then $f_j m_{p(j1)} = m_{q(j1)}$ and $f_j m_{p(j2)} = \lambda_j m_{q(j2)}$, where λ_j is the parameter of a double morphism f_j.

By changes of the basis vectors

$$m_i = x_i m_i', \quad 0 \neq x_i \in k, \tag{3}$$

we obtain a new set of double morphism: $f_j' = x_{p(j1)} x_{q(j1)}^{-1} f_j$, $1 \le j \le s$, with the parameters $\lambda_j' = \lambda_j x_{p(j1)} x_{q(j1)}^{-1} x_{p(j2)}^{-1} x_{q(j2)}$.

The change (3) gives a multiplicative basis if $\lambda_1' = \lambda_2' = ... = \lambda_s' = 1$, i.e. if $x_1, x_2, ..., x_r$ satisfy the system of equations

$$\lambda_j x_{p(j1)} x_{p(j2)}^{-1} = x_{q(j1)} x_{q(j2)}^{-1}, \quad 1 \le j \le s. \tag{4}$$

We shall solve the system by elimination: solve the first equation for some x_i and substitute the result in other equations. This amounts to the multiplication of each of them by rational power of the first equation. Further we solve the second equation of the obtained system for some x_j and substitute the result in other equations... There are two possibilities:

1. After the sth step, we obtain the solution $(x_1, ..., x_t) \in (k \setminus \{0\})^t$ of (4).

2. After the $(t-1)$th step $(1 \le t \le s)$, we obtain a system, the tth equation of which does not contain unknowns. In this case, the tth equation of (4), up to scalar multiples λ_t, is the product of rational powers of the 1th,...,$(t-1)$th equations. It means that there exist integers $z_1, ...z_t$ such that $z_t \neq 0$ and the equality

$$(x_{p(11)} x_{p(12)}^{-1})^{z_1} ... (x_{p(t1)} x_{p(t2)}^{-1})^{z_t} = (x_{q(11)} x_{q(12)}^{-1})^{z_1} ... (x_{q(t1)} x_{q(t2)}^{-1})^{z_t} \tag{5}$$

is the identity, i.e. each x_i has the same exponents at the two sides of (5).

Define the integer function $z : \Gamma_1 \to$Z by $z(f_{j1}) = -z(f_{j2}) = z_j$ for $j \le t$ and $z(f_{j1}) = z(f_{j2}) = 0$ for $j > t$. Since x_i corresponds to the vertex a_i of Γ, we have by (5) that this function is a non-zero weight function, which contradicts Lemma 13. Hence case 2 is impossible. This finishes the proof of Proposition 2.

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